Pointwise Circumscription Revisited

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Abstract

A decade ago, Pointwise Circumscription was proposed as a tool for formalizing common sense. In this paper, we revisit some of its uses and examine some cases in which it does not yield the intended behavior. Specifically, we explore the cases in which unsatisfiability may result from the presence of multiple minimal models (for McCarthy's Circumscription). Then, we present a form of circumscription, called *Point-Sensitive Circumscription*, that is a generalization of McCarthy's Circumscription and Lifschitz's Generalized Pointwise Circumscription. We illustrate how Point-Sensitive Circumscription handles these cases without losing the control over selective fine-grained variance of predicates and functions. Last, we compare the two tools and their potential uses in formalizing Theories of Action.

1 INTRODUCTION

Pointwise Circumscription, devised by Lifschitz in (Lifschitz, 1987) and (Lifschitz, 1989), is a nonmonotonic method defined along the intuitions of Circumscription (McCarthy, 1986) with superior control of the minimization process. Since its debut, it has been used in formalizing Common Sense problems and solutions, such as in (Doherty and Lukaszewicz, 1994), formalizing some Entailment Classes in the theory of *Features and Fluents* (cf (Sandewall, 1994)), and in (Amir, 1997), using Pointwise Circumscription to formalize the system of (Lin and Reiter, 1994). It was argued in (Lifschitz, 1987) and (Doherty and Lukaszewicz, 1994) that Pointwise Circumscription has the power to be a tool for formalizing solutions for the Frame Problem.

In this paper, we show that there are cases in which Pointwise Circumscription requires the intended model to be a minimum, rather than merely minimal (sections 2,3). We present a form of circumscription, called *Point-Sensitive Circumscription*, that is a generalization of McCarthy's Circumscription and Lifschitz's Generalized Pointwise Circumscription (sections 4,5.1). Point-Sensitive Circumscription allows us to get a consistent theory where Pointwise Circumscription fails to do so, while maintaining similar control over the minimization process. For Point-Sensitive Circumscription, we allow a varied range of point-wise-ness (the removal of elements of the minimized predicate separately), but maintain the same control over selective fine-grained variance of predicates and functions. We give semantics for our approach and demonstrate the different behavior using examples from the literature (section 5).

In this paper, we use the following two conventions: (1) the relation $p < q$ between two propositions refers to the usual ordering of Boolean values ($false < true$), with $p \leq q$ allowing $p = q$; and (2) the relation $P < Q$ between two predicates with same arity refers to the strict subset relation (regarding P, Q as the relative sets of elements).

2 POINTWISE CIRCUMSCRIPTION

2.1 CIRCUMSCRIPTION

McCarthy's Circumscription (McCarthy, 1980) is one of the first and major nonmonotonic reasoning tools. Since its debut, the nonmonotonic reasoning line of work has expanded and several textbooks now exist that give a fair view of nonmonotonic reasoning and its uses (e.g., (Brewka, 1991), (Antoniou, 1997), (Brewka et al., 1997), (D.M. Gabbay, 1994), (Sandewall, 1994), (Shanahan, 1997)). The motivations for nonmonotonic reasoning vary from formalizing Common Sense reasoning through Elaboration Tolerance and representation of uncertainty to Belief Revision. We do not expand on these motivations further here, but section 5 gives an example of the Com-

mon Sense application, and the reader may look at (Shanahan, 1997),(McCarthy, 1998),(Pearl, 1990) and (Antoniou, 1997) for further details in these directions.

McCarthy's Circumscription formula (McCarthy, 1980)

$$Circ[A(P,Z);P;Z] = \\ A(P,Z) \wedge \forall p,z\,(A(p,z) \implies \neg(p < P))$$

says that in the theory A, with parameter relations and function sequences P, Z, P is a minimal element such that $A(P,Z)$ is still consistent, when we are allowed to vary Z in order to allow P to become smaller.

Take for example the following simple theory:

$$T \equiv block(B_1) \wedge block(B_2)$$

Then, the circumscription of $block$ in T, varying nothing, is $Circ[T;block;] = T \wedge \forall p\,[T_{[block/p]} \implies \neg(p < block)]$ which is equivalent to

$$Circ[T;block;] \equiv \forall x\,(block(x) \Leftrightarrow (x = B_1 \vee x = B_2))$$

By minimizing $block$ we concluded that there are no other blocks in the world other than those mentioned in the original theory T.

2.2 POINTWISE CIRCUMSCRIPTION: THE FORMULA

Pointwise Circumscription was first proposed in (Lifschitz, 1986) and then expanded in (Lifschitz, 1987) and (Lifschitz, 1989). In (Doherty and Lukaszewicz, 1994) it was demonstrated that Pointwise Circumscription can be used to formalize various logics of action[1] in the framework of Features and Fluents (Sandewall, 1994). These logics include OCM (Original Chronological Minimization), PCM (Prototypical Chronological Minimization), PCMF (PCM with Filtering), CMON (Chronological Minimization of Occlusion with Nochange Premises) and CMOC (Chronological Minimization of Occlusion and Change).

The general idea behind Pointwise Circumscription is to capture the minimization idea of *Circumscription* (McCarthy, 1980), (McCarthy, 1986), while making the circumscription on different subsets of the domain for different stages of the minimization process.

The basic case for Pointwise Circumscription is the formula

$$A(P) \wedge \forall x \neg[Px \wedge A(\lambda y(Py \wedge x \neq y))]$$

(where we minimize the subscript P again, varying nothing), intuitively saying that we remove one element at a time. One of the benefits of such an approach is the first-orderedness of the circumscription formula. This property disappears, though, in the most general form, which we use below.

For Pointwise Circumscription, let $A(S_1,...,S_n)$ be a sentence in which each S_i is a predicate symbol or a function symbol (in particular, it can be a 0-ary function symbol, i.e., an object constant). We want to minimize one of the predicate symbols from this list, S_{i_0} (Thus, S_{i_0} corresponds to P, and the other members of the list correspond to Z in the notation used above). Let us write $EQ_V(P,Q)$ ("P and Q are equal outside V") for

$$EQ_V(P,Q) \stackrel{def}{=} \forall x(\neg Vx \implies (Px \equiv Qx)) \qquad (1)$$

for P, Q predicates or functions, and V a predicate, all with same arity. The *Pointwise circumscription* of S_{i_0} in A with S_i allowed to vary on V_i is, by definition,

$$A(S) \wedge \forall xs\neg[S_{i_0}x \wedge \neg s_{i_0}x \wedge \bigwedge_{i=1}^{n} EQ_{V_i x}(s_i,S_i) \wedge A(s)] \qquad (2)$$

Here, S stands for $S_1,...,S_n$, s is a list $s_1,...,s_n$ of predicate and function variables corresponding to the predicate and function constants S, and $\lambda x u V_i(x,u)$ $(i = 1,...,n)$ is a predicate without parameters that does not contain $S_1,...,S_n$ and whose arity is the arity of S_{i_0} plus the arity of S_i. We denote (2) by $C_{PW}[A;S_{i_0};S_1/V_1,...,S_n/V_n]$.

2.2.1 An Intuitive Explanation of Pointwise Circumscription

The Pointwise Circumscription formula (2) has the property that we remove x from S_{i_0} while varying S (recall that S is the list of predicates/functions $S_1,...,S_n$) in each of the $V_i x$ separately, fixing all the other parts of the domain, at that time.

In other words, the bracketed part of the formula says that, in the process of removing x from S_{i_0}, we allow only some parts of S to change. These parts are determined by $V_i x$ by saying that only where $V_i x$ is TRUE s_i may be different from S_i (recall that $EQ_{V_i x}(s_i,S_i)$ means that s_i is equal to S_i outside $\lambda u V_i(x,u)$).

2.3 SEMANTICS FOR POINTWISE CIRCUMSCRIPTION

In (Lifschitz, 1987), Pointwise Circumscription was given semantic interpretation.

Take, without loss of generality, $i_0 = 1$. For a model $\mathcal{M}$ of $A(S)$, let $|\mathcal{M}|$ be the associated universe, and for every term, function or predicate a, $a^{\mathcal{M}}$ is the realization of a in $\mathcal{M}$.

[1] Some of the formalizations needed *filtering*.

Definition 2.1 *Let $\mathcal{M}_1, \mathcal{M}_2$ have the same universe U, and let $\xi \in U^k$, where k is the arity of S_1. We say that $\mathcal{M}_1 \leq^{\xi} \mathcal{M}_2$ iff*

1. *$K^{\mathcal{M}_1} = K^{\mathcal{M}_2}$ for every function or predicate constant K that is not in S,*
2. *for any $i = 1, ..., n$, $S_i^{\mathcal{M}_1}$ and $S_i^{\mathcal{M}_2}$ coincide on $\{\eta \mid \neg V_i^{\mathcal{M}_1}(\xi, \eta)\}$*
3. *$S_1^{\mathcal{M}_1}(\xi) \implies S_1^{\mathcal{M}_2}(\xi)$.*

Proposition 2.2 ((Lifschitz, 1987)) *Let $\mathcal{M}$ be a model of $A(S)$.*

$$\mathcal{M} \models C_{PW}[A; S_1; S_1/V_1, ..., S_n/V_n] \iff \forall \mathcal{M}' \in [\![A(S)]\!]\, \forall \xi \in |\mathcal{M}|^k \neg(\mathcal{M}' \leq^{\xi} \mathcal{M} \wedge \mathcal{M} \not\leq^{\xi} \mathcal{M}')$$

Other words, $\mathcal{M} \models C_{PW}[A; S_1; S_1/V_1, ..., S_n/V_n]$ iff for each $\xi \in |\mathcal{M}|^k$, $\mathcal{M}$ is minimal relative to $\leq^{\xi}$.

3 A COUNTER-INTUITIVE EXAMPLE

In this section, we discuss only one simple example.

Let $A(P) = (P(X) \vee P(Y)) \wedge X \neq Y$. For this simple language (the only predicate is P, and there are exactly two object constant symbols, X, Y), we get the Pointwise Circumscription formula

$$A(P) \wedge \forall xp\neg[Px \wedge \neg px \wedge EQ_{V_P x}(p, P) \wedge A(p)].$$

Assume that for all x, $(V_P x) \equiv True$, so that p is allowed to differ from P with no limitations. We get the further simplified Pointwise Circumscription formula

$$A(P) \wedge \forall xp\neg[Px \wedge \neg px \wedge A(p)].$$

Let $C = C_{PW}[(P(X) \vee P(Y)) \wedge X \neq Y; P; P/\lambda x.True]$.

Proposition 3.1 $C \models FALSE$.

PROOF

$$\begin{array}{l} C_{PW}[A(P); P; P/\lambda x.True] \equiv \\ A(P) \wedge \forall xp\neg[P(x) \wedge \neg p(x) \wedge A(p)] \equiv \\ A(P) \wedge \\ \quad \forall xp\neg[P(x) \wedge \neg p(x) \wedge (p(X) \vee p(Y)) \wedge X \neq Y] \\ \quad \implies \\ A(P) \wedge \\ \quad \forall p\neg \left[\begin{array}{l} P(X) \wedge \neg p(X) \wedge (p(X) \vee p(Y)) \wedge X \neq Y \vee \\ P(Y) \wedge \neg p(Y) \wedge (p(X) \vee p(Y)) \wedge X \neq Y \end{array} \right] \\ \equiv \\ A(P) \wedge \forall p\neg \left[\begin{array}{l} P(X) \wedge \neg p(X) \wedge p(Y) \wedge X \neq Y \vee \\ P(Y) \wedge \neg p(Y) \wedge p(X) \wedge X \neq Y \end{array} \right] \end{array}$$

Let $p_x = \{X\}, p_y = \{Y\}$ be two possible values of p. Then, from the formula on the last line, we get

$$\begin{array}{l} A(P) \wedge \forall p\neg \left[\begin{array}{l} P(X) \wedge \neg p(X) \wedge p(Y) \wedge X \neq Y \vee \\ P(Y) \wedge \neg p(Y) \wedge p(X) \wedge X \neq Y \end{array} \right] \\ \quad \implies \\ A(P) \wedge \neg \left[\begin{array}{l} P(X) \wedge \neg p_x(X) \wedge p_x(Y) \wedge X \neq Y \vee \\ P(Y) \wedge \neg p_x(Y) \wedge p_x(X) \wedge X \neq Y \vee \\ P(X) \wedge \neg p_y(X) \wedge p_y(Y) \wedge X \neq Y \vee \\ P(Y) \wedge \neg p_y(Y) \wedge p_y(X) \wedge X \neq Y \end{array} \right] \\ \equiv \\ A(P) \wedge \\ \quad \neg \left[\begin{array}{l} FALSE \vee \\ P(Y) \wedge (TRUE \iff X \neq Y) \wedge X \neq Y \vee \\ P(X) \wedge (TRUE \iff X \neq Y) \wedge X \neq Y \vee \\ FALSE \end{array} \right] \\ \equiv \\ A(P) \wedge \neg[(P(Y) \vee P(X)) \wedge X \neq Y] \equiv \\ P(X) \vee P(Y) \wedge X \neq Y \wedge \\ \quad \neg[(P(Y) \vee P(X)) \wedge X \neq Y] \equiv \\ FALSE \end{array}$$

■

This proposition reveals a limitation of Pointwise Circumscription, i.e., the requirement, in some cases, that P be a *minimum* rather than *minimal*[2].

This conclusion is supported by the semantics given by Lifschitz (proposition 2.2 above). Let $U = \{x, y\}$ be the set of elements in the universe. Let M_X, M_Y, M_{XY}, $M_\emptyset$ be the models with universe U, with X, Y interpreted to x, y, respectively, and the following interpretations for the predicate P:

$$\begin{array}{ll} P^{M_X} = \{X\} & P^{M_Y} = \{Y\} \\ P^{M_{XY}} = \{XY\} & P^{M_\emptyset} = \emptyset \end{array}$$

Figure 1 below displays the different models for $A(P)$.

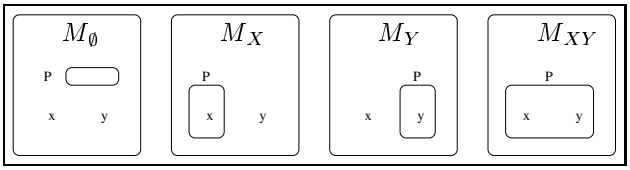

Figure 1: The four models of $A(P)$ with universe $U = \{x, y\}$.

There are exactly two elements in U, and thus there are two partial orders $\leq^x$,$\leq^y$ between the models. In order to check these partial orders, we first notice that the first two

[2]Here *minimum* refers to an object that is smaller (in terms of the given partial order relation) than all other objects, while *minimal* refers to objects for which there is no smaller object.

conditions of definition 2.1 are met by any pair of models from the four examined above. Using the third condition, we get the obvious $M_\emptyset \leq^y M_X \leq^y M_{XY}$, $M_{XY} \not\leq^y M_X$, $M_\emptyset \leq^x M_Y \leq^x M_{XY}$ and $M_{XY} \not\leq^x M_Y$. We also get the somewhat less obvious $M_\emptyset \leq^x M_Y \leq^x M_X$, $M_X \not\leq^x M_Y$, $M_\emptyset \leq^y M_X \leq^y M_Y$ and $M_Y \not\leq^y M_X$.

Examining the conditions of proposition 2.2 for each one of the models, we reveal the following:

- $M_\emptyset \not\models C$ because $M_\emptyset \not\models A(P)$.
- $M_X \not\models C$ because $M_Y \models A(P)$ and $M_Y \leq^x M_X$ and $M_X \not\leq^x M_Y$.
- $M_Y \not\models C$ because $M_X \models A(P)$ and $M_X \leq^y M_Y$ and $M_Y \not\leq^y M_X$.
- $M_{XY} \not\models C$ because $M_X \models A(P)$ and $M_X \leq^x M_{XY}$ and $M_{XY} \not\leq^y M_X$.

Thus, again, there is no model that satisfies C. Figure 2 below displays the two orders on models.

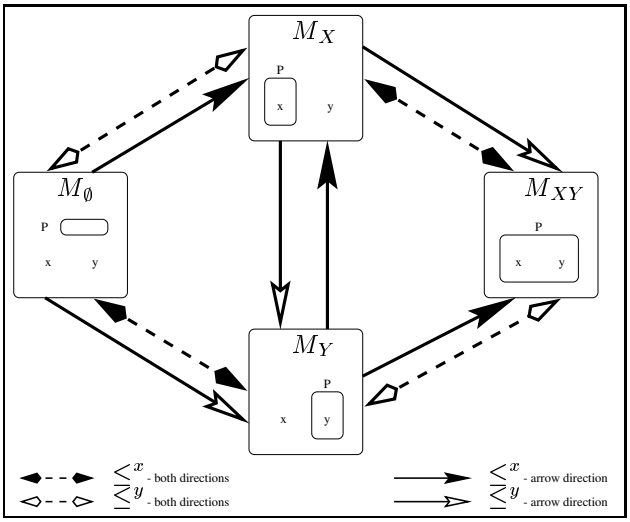

Figure 2: The orders $\leq^x$ and $\leq^y$. The different colors represent the different orders.

4 POINT-SENSITIVE CIRCUMSCRIPTION

4.1 THE FORMULA

We present *Point-Sensitive Circumscription*, a modified version of Pointwise Circumscription in which the minimized predicate is minimized according to a *minimization domain*. This minimization domain may be a point and may be the complete set of elements. We preserve the ability to *select/vary* parts of the theory/domain dynamically.

It is important to notice at this point that Lifschitz already proposed a global circumscription that has some control over the varied domain (see formula (14) in (Lifschitz, 1987)). We shall compare that proposal with Pointwise Circumscription and Point-Sensitive Circumscription in section 5.

We use similar notations to those used in section 2. Let $A(S_1, ..., S_n)$ be a sentence in which each S_i is a predicate symbol or a function symbol (in particular, it can be a 0-ary function symbol, i.e., an object constant). We want to minimize one of the *predicate* symbols from this list, S_{i_0}.

In addition to the definition of EQ_V in (1), let us write $LS_R(P,Q)$ ("P is smaller than Q in the region R", or $P \cap R \subsetneq Q \cap R$) for

$$LS_R(P,Q) \stackrel{def}{=} \forall x(Rx \Rightarrow (Px \Rightarrow Qx)) \wedge \exists x(Rx \wedge \neg Px \wedge Qx) \tag{3}$$

for P, Q predicates or functions, and R a predicate, all with same arity. The simplest example is where R is uniformly true, in which case $LS_{Rx}(P,Q) \equiv P \subsetneq Q$.

The *Point-Sensitive Circumscription* of S_{i_0} in A with S_i allowed to vary on V_i, and S_{i_0} minimized using R, denoted $\mathcal{C}_{PS}[A; S_{i_0}/R; S_1/V_1, ..., S_n/V_n]$, is, by definition,

$$A(S) \wedge \forall xs\neg[LS_{Rx}(s_{i_0}, S_{i_0}) \wedge \bigwedge_{i=1}^{n} EQ_{V_i x}(s_i, S_i) \wedge A(s)] \tag{4}$$

where S stands for $S_1, ..., S_n$, s is a list $s_1, ..., s_n$ of predicate and function variables corresponding to the predicate and function constants S, and $\lambda xuR(x,u)$, $\lambda xuV_i(x,u)$ $(i = 1, ..., n)$ are predicates without parameters that do not contain $S_1, ..., S_n$ and whose arity is the arity of S_{i_0} plus the arity of S_{i_0} and S_i, respectively (here $Rx = \lambda uR(x,u)$, $V_i x = \lambda u V_i(x,u)$).

4.1.1 An Intuitive Explanation

The bracketed part of the Point-Sensitive Circumscription formula (4) says that, in the process of making S_{i_0} smaller, we use R to define what "smaller" means and we allow only some parts of S to change. These parts are determined by Rx and $V_i x$ by saying that s_i may be different from S_i only where $V_i x$ is TRUE (recall that $EQ_{V_i x}(s_i, S_i)$ means that s_i is equal to S_i outside $\lambda u(V_i(x,u))$) and that we actually care if s_{i_0} is smaller only where Rx is TRUE.

Notice that the only difference between the Point-Sensitive Circumscription formula (4) and the Pointwise Circumscription formula (2) is in the first component of the bracketed formula. In that part, we changed the term $S_{i_0} x \wedge \neg s_{i_0} x$ (in (2)) to $LS_{Rx}(s_{i_0}, S_{i_0})$ (in (4)).

It is also important to notice that even for the cases where R is uniformly TRUE (in these cases, x is used only in the second component in the formula in parentheses), the second component is actually a conjunction of formulas, all using x. Therefore, although we do not require that x be removed from the predicate S_{i_0}, we do require that the variance of the different predicates/functions (including S_{i_0}) be correlated with x as an index.

4.2 SEMANTICS FOR POINT-SENSITIVE CIRCUMSCRIPTION

We follow the lines of (Lifschitz, 1987) with regard to the semantics for our Point-Sensitive Circumscription.

Take, without loss of generality, $i_0 = 1$. For a model $\mathcal{M}$ of $A(S)$, let $|\mathcal{M}|$ be the associated universe and, for every term, function or predicate a, $a^{\mathcal{M}}$ is the realization of a in $\mathcal{M}$. We write the corresponding definition to 2.1.

Definition 4.1 *Let $\mathcal{M}_1, \mathcal{M}_2$ have the same universe U, and let $\xi \in U^k$, where k is the arity of S_1. We say that $\mathcal{M}_1 \ll^{\xi} \mathcal{M}_2$ (a strict partial order) iff*

1. *$K^{\mathcal{M}_1} = K^{\mathcal{M}_2}$ for every function or predicate constant K that is not in S,*
2. *for any $i = 1, ..., n$, $S_i^{\mathcal{M}_1}$ and $S_i^{\mathcal{M}_2}$ coincide on $\{\eta \mid \neg V_i^{\mathcal{M}_1}(\xi, \eta)\}$*
3. *$LS_{R(\xi)}(S_1^{\mathcal{M}_1}, S_1^{\mathcal{M}_2})$ ($R(\xi) = \lambda u R(\xi, u)$).*

Let $[\![A(S)]\!]$ be the set of models of $A(S)$. The following proposition says that every model of the circumscription formula for $A(S)$ is minimal in $[\![A(S)]\!]$ according to all of the orders $\ll^{\xi}$, and vice versa.

Proposition 4.2 (Semantics) *Let $\mathcal{M} \in [\![A(S)]\!]$.*

$$\mathcal{M} \models C_{PS}[A; S_1/R; S_1/V_1, ..., S_n/V_n] \iff \forall \mathcal{M}' \in [\![A(S)]\!]\ \forall \xi \in |\mathcal{M}|^k\ \neg(\mathcal{M}' \ll^{\xi} \mathcal{M})$$

PROOF For the forward direction, let us take $\mathcal{M} \models \mathcal{C}_{PS}[A; S_1/R; S_1/V_1, ..., S_n/V_n]$. Assume that $\mathcal{M}' \in [\![A(S)]\!]$ $\mathcal{M}' \ll^{\xi} \mathcal{M}$ for some $\xi \in |\mathcal{M}|^k$. Then $LS_{R(\xi)}(S_1^{\mathcal{M}'}, S_1^{\mathcal{M}})$, by definition 4.1 (requirement 3).

Take s to be $S^{\mathcal{M}'}$, the sequence of predicates/functions interpreting S (the sequence of predicate/function constant symbols) in $\mathcal{M}'$. For $x = \xi$ (ξ picked above), we get that the following are true in the model $\mathcal{M}$:

- $LS_{R(\xi)}(s_1, S_1)$, because $s = (s_1, ..., s_n)$ (that is our notation), and the way we picked ξ (s.t., $s_1 = S_1^{\mathcal{M}'}$ and $LS_{R(\xi)}(S_1^{\mathcal{M}'}, S_1^{\mathcal{M}})$),
- $A(s)$, because $s = S^{\mathcal{M}'}$ and $M' \models A(S)$, and
- $\bigwedge_{i=1}^{n} EQ_{V_i x}(s_i, S_i)$, because of definition 4.1 (requirement 2).

This contradicts $\mathcal{M} \models \mathcal{C}_{PS}[A; S_1/R; S_1/V_1, ..., S_n/V_n]$, because we found x and s that satisfy the bracketed part of formula (4).

The reverse direction works by the same method. If $\mathcal{M}$ is $\ll^{\xi}$-minimal for every $\xi \in |\mathcal{M}|^k$, but does not satisfy formula (4), then there are s, x such that $LS_{R(\xi)}(s_1, S_1) \wedge \bigwedge_{i=1}^{n} EQ_{V_i x}(s_i, S_i) \wedge A(s)$. But then we can build a model $\mathcal{M}'$ with the same universe, taking all the constants other than those in S to be the same, and the constants of S take the values of s.

For $\mathcal{M}'$, we show now that $\mathcal{M}' \ll^{x} \mathcal{M}$. We follow the conditions of definition 4.1 one by one.

1. is satisfied by the construction of $\mathcal{M}'$.
2. is satisfied because s satisfies the second component of the bracketed part of the C_{PS} formula.
3. is satisfied because s satisfies the first component of the bracketed part of the C_{PS} formula.

We have found a model $\mathcal{M}'$ with $\mathcal{M}' \ll^{x} \mathcal{M}$. Contradiction. ■

Lemma 4.3 *For $\xi \in |\mathcal{M}|^k$, $\ll^{\xi}$ is a strict partial order over models of $\mathcal{L}$.*

PROOF Irreflexivity is simply because $LS_R(P, Q)$ is irreflexive ($R \cap P \subsetneq Q \cap R$). It is now enough to prove transitivity. Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ with same domain, and $\xi \in |\mathcal{M}_1|^k$. Assume $\mathcal{M}_1 \leq^{\xi} \mathcal{M}_2$ and $\mathcal{M}_2 \leq^{\xi} \mathcal{M}_3$. We prove the conditions for $\mathcal{M}_1 \leq^{\xi} \mathcal{M}_3$ one by one.

1. For K not in S, $K^{\mathcal{M}_1} = K^{\mathcal{M}_2} = K^{\mathcal{M}_3}$.
2. For every $i \in \{1, ..., n\}$, $S_i^{\mathcal{M}_1}, S_i^{\mathcal{M}_2}$ coincide on $\{\eta \mid \neg V_i^{\mathcal{M}_1}(\xi, \eta)\}$, and $S_i^{\mathcal{M}_2}, S_i^{\mathcal{M}_3}$ coincide on $\{\eta \mid \neg V_i^{\mathcal{M}_2}(\xi, \eta)\}$. Since $V_i^{\mathcal{M}_2} = V_i^{\mathcal{M}_1}$ (recall that $\lambda x, u V_i(x, u)$ is a formula that contains no element from S), $S_i^{\mathcal{M}_2}, S_i^{\mathcal{M}_3}$ coincide on $\{\eta \mid \neg V_i^{\mathcal{M}_1}(\xi, \eta)\}$. Thus $S_i^{\mathcal{M}_1}, S_i^{\mathcal{M}_3}$ coincide on $\{\eta \mid \neg V_i^{\mathcal{M}_1}(\xi, \eta)\}$.
3. $LS_{R(\xi)}(S_1^{\mathcal{M}_1}, S_1^{\mathcal{M}_2})$ and $LS_{R(\xi)}(S_1^{\mathcal{M}_2}, S_1^{\mathcal{M}_3})$, and thus (transitivity of $\subsetneq$) $LS_{R(\xi)}(S_1^{\mathcal{M}_1}, S_1^{\mathcal{M}_3})$.

Thus we proved all three requirements for $\mathcal{M}_1 \ll^{\xi} \mathcal{M}_3$ and transitivity is proved. Assymetry follows from transitivity and irreflexivity. ■

Recall that our notation is $Rx = \lambda u R(x,u)$ and that R is used to control the domain for the comparison of the minimized predicates. Intuitively, the following proposition says that if we always restrict the comparison of the minimized predicates to the same domain and there are finitely many models to the original theory, then the circumscription is satisfiable.

Proposition 4.4 *Let A be a satisfiable theory with finitely many models, and let $Rx = Ry$ for all $x, y \in |\mathcal{M}|^k$. Then, $\mathcal{C}_{PS}[A; S_1/R; S_1/V_1, ..., S_n/V_n]$ is satisfiable.*

PROOF Assume there is no model $\mathcal{M}$ that satisfies $\mathcal{C}_{PS}[A; S_1/R; S_1/V_1, ..., S_n/V_n]$. Then, there is no model $\mathcal{M}$ in $[\![A(S)]\!]$ such that $\forall \mathcal{M}' \in [\![A(S)]\!]\ \forall \xi \in |\mathcal{M}|^k\ \neg(\mathcal{M}' \ll^{\xi} \mathcal{M})$. Take $\xi_0 \in |\mathcal{M}|^k$. Since $[\![A]\!]$ is finite and $\ll^{\xi_0}$ is a strict partial order, there must be at least one minimal element $\mathcal{M}$ for $\ll^{\xi_0}$ from the models of A. If a minimal model $\mathcal{M}_1$ of $\ll^{\xi_0}$ is not a minimal element according to some $\ll^{\xi_1}$, then $\ll^{\xi_1}$ has a minimal model $\mathcal{M}_2 \ll^{\xi_1} \mathcal{M}_1$. Since $\mathcal{M}_2$ is not minimal according to all of the $\ll$'s, there is yet another model $\mathcal{M}_3$ that is smaller according to yet another $\ll^{\xi_2}$.

Since there are finitely many models, if we continue such a chain of models, we will eventually end up with a cycle. Without loss of generality assume that the cycle is $\mathcal{M}_1, \mathcal{M}_2, ..., \mathcal{M}_n, \mathcal{M}_1$. But then we get $LS_{R(\xi_n)}(S_1^{\mathcal{M}_1}, S_1^{\mathcal{M}_n}), ... LS_{R(\xi_1)}(S_1^{\mathcal{M}_2}, S_1^{\mathcal{M}_1})$. Now we use the fact that $R(\xi_1) \equiv R(\xi_2) \equiv ... \equiv R(\xi_n)$ and get that $LS_{R(\xi_1)}(S_1^{\mathcal{M}_1}, S_1^{\mathcal{M}_n}), ... LS_{R(\xi_1)}(S_1^{\mathcal{M}_2}, S_1^{\mathcal{M}_1})$. Therefore, using the transitivity of $LS_{R(\xi_1)}$ we get $LS_{R(\xi_1)}(S_1^{\mathcal{M}_1}, S_1^{\mathcal{M}_1})$, contradicting the irreflexivity of $LS_{R(\xi_1)}$. ■

Last, we find out the following proposition.

Proposition 4.5 *1. McCarthy's Circumscription is equivalent to Point-Sensitive Circumscription with $R \equiv TRUE$ and $V \equiv TRUE$.*

2. Lifschitz's Generalized Pointwise Circumscription is equivalent to Point-Sensitive Circumscription with $Rx \equiv \lambda x'(x = x')$.

PROOF

1. $\mathcal{C}_{PS}[A; S_1/TRUE; S_1/TRUE, ..., S_n/TRUE] = A(S) \wedge \forall s \neg[LS_{TRUE}(s_1, S_1) \wedge \bigwedge_{i=1}^{n} EQ_{TRUE}(s_i, S_i) \wedge A(s)]$ s_1 is allowed to differ from S_1 with no limitations. Since $LS_{TRUE}(s_1, S_1) \equiv (s_1 \subsetneq S_1)$ we get the simplified Point-Sensitive Circumscription formula

$$A(S) \wedge \forall s \neg[s_1 < S_1 \wedge A(S)]$$

which is exactly McCarthy's Circumscription formula.

2. $\mathcal{C}_{PS}[A; S_1/Rx; S_1/V_1, ..., S_n/V_n]$ is, by definition, $A(S) \wedge \forall s \neg[LS_{Rx}(s_1, S_1) \wedge \bigwedge_{i=1}^{n} EQ_{Vix}(s_i, S_i) \wedge A(s)]$. Let $Rx \equiv \lambda x'(x = x')$. Then, $LS_{Rx}(P, Q)$ is $\forall x'(Rx(x') \Rightarrow (Px' \Rightarrow Qx')) \wedge \exists x'(Rx(x') \wedge \neg Px' \wedge Qx')$ which is equivalent to $\neg Px \wedge Qx$. Thus, we get $\mathcal{C}_{PS}[A; S_1/Rx; S_1/V_1, ..., S_n/V_n] \equiv C_{PW}[A; S_{i_0}; S_1/V_1, ..., S_n/V_n]$. ■

5 EXAMPLES USING POINT-SENSITIVE CIRCUMSCRIPTION

We examine two applications of Point-Sensitive Circumscription. First, we shortly re-examine the example from section 3. Then, we review one application of Pointwise Circumscription for the formalization of Prototypical Chronological Minimization (PCM), which is one known solution to the Frame Problem (for some ontological classes[3]). Finally, we examine this application with the proposed Point-Sensitive Circumscription and with Generalized Global Circumscription (as proposed in (Lifschitz, 1987) (formula (14))).

5.1 MULTIPLE MINIMAL MODELS

Let us first recall the example from section 3. Let $A(P) = (P(X) \vee P(Y)) \wedge X \neq Y$. Let $(V_P x) \equiv TRUE$ for all x, so that p is allowed to differ from P with no limitations. Let $R \equiv TRUE$, so that $LS_{Rx}(P, Q) \equiv (P \subsetneq Q)$. We get the simplified Point-Sensitive Circumscription formula

$$A(P) \wedge \forall p \neg[p < P \wedge A(p)]$$

which is exactly McCarthy's Circumscription formula. Thus, for $A(P) = (P(X) \vee P(Y)) \wedge X \neq Y$, we get the following result (contrary to proposition 3.1 for the Pointwise Circumscription case):

Proposition 5.1 *$\mathcal{C}_{PS}[A(P); P/R; P/\lambda x.True]$ is equivalent to*

$$(\forall x(P(x) \Leftrightarrow x = X)) \vee (\forall x(P(x) \Leftrightarrow x = Y)) \wedge X \neq Y$$

.

PROOF First, notice that By theorem 4.5, $\mathcal{C}_{PS}[A(P); P; P/\lambda x.True]$ is in fact equivalent to $A(P) \wedge \forall p \neg[p < P \wedge A(p)]$. At this point, all we need to notice is that we got McCarthy's Circumscription which is known to have the required equivalence. ■

Let us now re-examine the situation as seen by the semantics of proposition 4.2. Let $U = \{x, y\}$ be the set of el-

[3] See (Sandewall, 1994).

ements in the universe. We explore the same models discussed in section 3, M_X, M_Y, M_{XY} and $M_\emptyset$. Figure 3 below displays the two orders $\ll^X, \ll^Y$ on models.

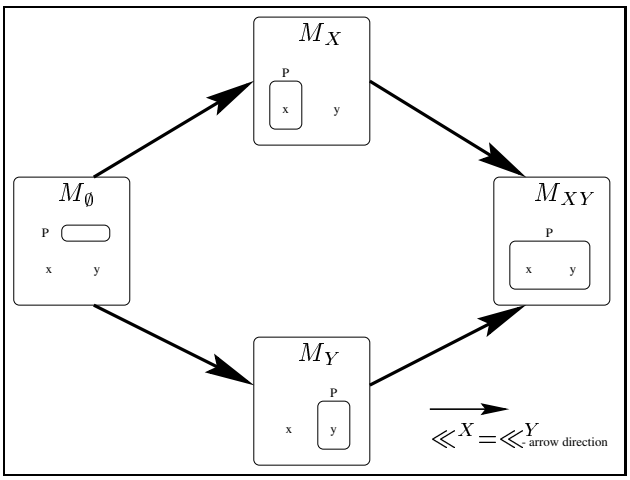

Figure 3: The orders $\ll^X$ and $\ll^Y$.

5.2 PCM USING POINTWISE CIRCUMSCRIPTION

Doherty and Lukaszewicz (Doherty and Lukaszewicz, 1994) examined various solutions to the Frame Problem, implementing them using either McCarthy's Circumscription or Lifschitz's Generalized Pointwise Circumscription (formula (2) above). We briefly review some of the theory of Features and Fluents (Sandewall, 1994) and one of the solutions that was examined in (Doherty and Lukaszewicz, 1994), namely, PCM, for which Pointwise Circumscription was used.

5.2.1 The general theory for PCM

The language $\mathcal{L}(FL)$ is a sorted first-order language with equality. There are two domain independent sorts, $\mathcal{T}$ for time points (t, s are variables of that sort) and $\mathcal{F}$ for propositional fluents (f, g are variables of that sort), which are functions from time to truth values. We include the predicate symbols $Holds, Clip$ of type $\mathcal{T} \times \mathcal{F}$ and the predicate symbols $<, \leq$ of type $\mathcal{T} \times \mathcal{T}$ (total orders on time points).

Let $\Gamma_C = \Gamma_{OBS} \cup \Gamma_{SCD} \cup \Gamma_{UNA}$ denote the set of formulas in $\mathcal{L}(FL)$ corresponding to the given scenario description, where Γ_{OBS} are the observation axioms (e.g., $Holds(f, s)$), Γ_{SCD} are the schedule axioms (a form of "effect axioms") and Γ_{UNA} are the appropriate unique name axioms for the fluent constants. Γ_{PER} denotes the set containing the two persistence axioms

$$\forall ft.Holds(t, f) \Rightarrow [Holds(t+1, f) \bar{\vee} Clip(t+1, f)]$$
$$\forall ft.\neg Holds(t, f) \Rightarrow [\neg Holds(t+1, f) \bar{\vee} Clip(t+1, f)]$$

PCM is defined roughly as follows (see (Sandewall and Shoham, 1994) for more details): Interpretations $I \ll_{pcm} I'$ iff $|I| = |I'|$ (the objects are the same) and there is t_0 such that

1. for all $t < t_0$, $R(t) = R'(t)$ (the fluents are identical in I and I' before t_0) and
2. $breakset(I, t_0) \subset breakset(I', t_0)$ (the fluent-value changes from $t_0 - 1$ to t_0 in I are a subset of the appropriate changes in I').

Then the set of models for PCM is $S_pcm(T) = Min(\ll_{pcm}, [\![T]\!])$.

For the logic of PCM, we include the following. Let $\Gamma(Clip, Holds)$ denote the theory $\Gamma_C \cup \Gamma_{PER}$. Let $V(t, f) = \lambda t' f'.[\langle t, f\rangle = \langle t', f'\rangle \vee t < t']$ and $U(t, f) = \lambda t' f'.[t \leq t']$. Then, $C_{PW}[\Gamma(Clip, Holds); Clip; Clip/V, Holds/U]$ is

$$\begin{array}{l}\Gamma(Clip, Holds) \wedge \\ \forall t, f. \forall clip, holds. \neg [Clip(t, f) \wedge \neg clip(t, f) \wedge \\ \quad EQ_{V(t,f)}(Clip, clip) \wedge EQ_{U(t,f)}(Holds, holds) \wedge \\ \quad \Gamma(clip, holds)]\end{array} \tag{5}$$

The intention is that when minimizing $Clip$ at time t for fluent f, the only part of $Clip$'s extension that will vary is the point $\langle t, f\rangle$ itself or $\langle t', f'\rangle$ for $t' > t$ and arbitrary f'. In addition, $Holds$ is varied only for time points $t' \geq t$.

5.2.2 YSP: global minimization and nondeterministic actions

Let us examine a variant of the Yale Shooting Scenario (YSS) ((Hanks and McDermott, 1986)). Assume that there are two turkeys, Turkey1 and Turkey2. As a result of the gun's being shot, exactly one turkey dies. For this, we have the propositions ALIVE1, DEAD1, ALIVE2, DEAD2, LOADED. Initially, both turkeys are alive, and the gun is LOADED. We have the action sequence of waiting and then shooting.

$$\begin{array}{ll} obs1 & [0, \infty)(ALIVE1 \leftrightarrow \neg DEAD1) \\ obs1 & [0, \infty)(ALIVE2 \leftrightarrow \neg DEAD2) \\ obs3 & [0]ALIVE1, ALIVE2, LOADED \\ scd1 & [\mathbf{s}]LOADED \rightarrow [\mathbf{s}, \mathbf{t}]ALIVE1 := TF \\ obs4 & [\mathbf{s}]LOADED \rightarrow [\mathbf{s}, \mathbf{t}]ALIVE2 \leftrightarrow \neg ALIVE1 \\ scd2 & 0 < \mathbf{s} < \mathbf{t} \end{array} \tag{6}$$

$scd1$ says that, given that at time $\mathbf{s}$ the gun is loaded, then between $\mathbf{s}$ and $\mathbf{t}$, Turkey1 either dies or stays alive (as a result of the shooting action). $obs4$ then says that in the

process of executing the turkeys (the time frame $[\mathbf{s}, \mathbf{t}]$), one turkey stays alive iff the other dies[4] (notice that we had to write $obs4$ as an observation because of the cues of the language). One of the intended results of applying (5) is that the turkeys are still alive at time $\mathbf{s}$, but this is not what we get. Applying (5) yields two minimal models with

$$\begin{array}{lll} \mathcal{M}_1 \models & obs3 & [\mathbf{s}]ALIVE1 \\ \mathcal{M}_2 \models & obs3 & [\mathbf{s}]\neg ALIVE1 \end{array}$$

The reason for the unintended minimal model is the insistence on minimizing one fluent at a time, while not allowing other changes for other fluents for the same time point.

One proposed solution might be to minimize change for all the fluents at once for a given time point. For this solution, we set the policy V to be

$$V(t, f) = \lambda t' f'.[t \leq t']. \tag{7}$$

Trying to use (7), we now encounter the problem discussed in section 3, i.e., that in the presence of fluents that are minimized in a circular fashion (i.e., minimizing P while varying Q and vice versa), the minimal models must agree on the involved predicates.

There are two intended minimal models for this scenario, one in which Turkey1 is eventually dead, and another in which Turkey2 is eventually dead. But $\leq^{\langle \mathbf{t}, ALIVE1 \rangle}$ prefers Turkey2 to be dead, and $\leq^{\langle \mathbf{t}, \overline{ALIVE1} \rangle}$ prefers Turkey2 to be dead. Thus, neither is a minimal model, and Pointwise Circumscription incurs inconsistency.

5.3 COMPARING SOLUTIONS

It is important to note that the problems we describe above do not affect the findings of (Doherty and Lukaszewicz, 1994), because these problems fall outside the ontological class that the formalization of (Doherty and Lukaszewicz, 1994) tried to formalize, namely the ontological class $\mathcal{K}p\text{-}IAex$. This class assumes that there are no observations outside the initial state. For the purposes of (Doherty and Lukaszewicz, 1994), a domain constraint such as $DEAD1 \leftrightarrow \neg ALIVE1$ is an observation for time points later than 0. Thus, to include this more *advanced* class, we need *filtering*[5].

Using Point-Sensitive Circumscription, we can do better than that. Let the range for comparison be controlled by $R(t, f) = \lambda t' f'[t' = t]$. As a result of applying the circumscription policy (7) to theory (6), we get

$$\begin{array}{l} \mathcal{C}_{PS}[\Gamma(Clip, Holds); Clip/R; Clip/V, Holds/U] = \\ \quad \Gamma(Clip, Holds) \wedge \\ \quad \forall t, f \forall clip, holds \neg [LS_{R(t,f)}(clip, Clip) \wedge \\ \qquad EQ_{V(t,f)}(Clip, clip) \wedge EQ_{U(t,f)}(Holds, holds) \wedge \\ \qquad \Gamma(clip, holds)] \end{array} \tag{8}$$

This formula minimizes $Clip$ for each time point for all fluents together (rather than per fluent). We get the following two minimal models

$$\begin{array}{lll} \mathcal{M}_1 \models & obs5 & [\mathbf{t1}]\neg ALIVE1, DEAD1, \\ & & \neg ALIVE2, DEAD2, LOADED \\ \mathcal{M}_2 \models & obs5 & [\mathbf{t1}]ALIVE1, \neg DEAD1, \\ & & \neg ALIVE2, DEAD2, LOADED \end{array}$$

which is exactly the outcome we wanted.

We do not claim that Point-Sensitive Circumscription can formalize the entire $\mathcal{K}\text{-}IA$ ontological class, but that it can take you farther than Pointwise Circumscription can. In more precise terms, we showed that Point-Sensitive Circumscription can simulate PCM in some cases that are outside $\mathcal{K}p - IAex$ (the class treated by (Doherty and Lukaszewicz, 1994)). A hint that Point-Sensitive Circumscription may be able to simulate the ramification as given in the semantics of (Lin and Reiter, 1994), was given in (Amir, 1997).

Last, let us examine another proposal made in (Lifschitz, 1987), namely, an expanded version of *Global Circumscription*:

$$\begin{array}{l} Circ_G[A; P; Z/V] = \\ \quad A(P, Z) \wedge \forall pz \neg [EQ_V(z, Z) \wedge A(p, z) \wedge p < P] \end{array}$$

This is a reduced version of C_{PS} (taking $R \equiv TRUE$ $V(t, f) \equiv V$, we get $C_{PS} \equiv Circ_G$) in which we can not use different domains V_x for different points in the minimization. In the example above, it is essential to vary the fluents only for time points later than the current time point. Therefore, this proposal is not enough for the representation of the preference needed above.

6 CONCLUSIONS

In the context of Theories of Action, Pointwise Circumscription is sufficient if we restrict ourselves to deterministic actions and to cases in which there are no domain constraints and no action's effects are dependent on other actions' effects.

We showed that Pointwise Circumscription does not yield the right conclusions for some cases in which there are multiple minimal models (for McCarthy's Circumscription). This is due to the "point-wise-ness" of that form of circumscription. We described Point-Sensitive Circumscription,

[4] Meaning that exactly one turkey dies.

[5] Notice that our example (6) is in $\mathcal{K}\text{-}IAenx$.

a generalization of both McCarthy's Circumscription and Lifschitz's Pointwise Circumscription, and showed that it is useful for Theories of Action. Point-Sensitive Circumscription allows us to choose our degree of "point-wiseness" (as suggested by the fact that both Circumscription and Pointwise Circumscription are its special cases) by controlling the domain used for the comparison of predicates. Since most uses of Pointwise Circumscription to date (e.g., (Doherty and Lukaszewicz, 1994)) refer mostly to the last property (variance control) rather than the first one ("point-wise-ness"), Point-Sensitive Circumscription is a useful substitute.

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